Perturbations of Schwarzschild Black Holes in Chern-Simons Modified Gravity

Nicolás Yunes[1] and Carlos F. Sopuerta[2, 3, 4]

[1] *Center for Gravitational Wave Physics and Institute for Gravitational Physics and Geometry, Department of Physics, The Pennsylvania State University, University Park, PA 16802, USA*
[2] *Department of Physics, University of Guelph, Ontario N1G 2W1, Canada*
[3] *Institut de Ciències de l'Espai (CSIC), Facultat de Ciències, Campus UAB, Torre C5 parells, E-08193 Bellaterra, Spain*
[4] *Institut d'Estudis Espacials de Catalunya (IEEC), Ed. Nexus-201, Gran Capità 2-4, E-08034 Barcelona, Spain*

(Dated: February 2, 2008)

We study perturbations of a Schwarzschild black hole in Chern-Simons modified gravity. We begin by showing that Birkhoff's theorem holds for a wide family of Chern-Simons *coupling* functions, a scalar field present in the theory that controls the strength of the Chern-Simons correction to the Einstein-Hilbert action. After decomposing the perturbations in spherical harmonics, we study the linearized modified field equations and find that axial and polar modes are coupled, in contrast to general relativity. The divergence of the modified equations leads to the *Pontryagin constraint*, which forces the vanishing of the Cunningham-Price-Moncrief master function associated with axial modes. We analyze the structure of these equations and find that the appearance of the Pontryagin constraint yields an overconstrained system that does not allow for generic black hole oscillations. We illustrate this situation by studying the case characterized by a canonical choice of the coupling function and pure-parity perturbative modes. We end with a discussion of how to extend Chern-Simons modified gravity to bypass the Pontryagin constraint and the suppression of perturbations.

PACS numbers: 04.50.Kd,04.25.-g,04.25.Nk,04.30.-w

I. INTRODUCTION

Extensions of general relativity (GR) are inherently interesting because they hold the promise to address unresolved problems in cosmology and astrophysics. One such extension is Chern-Simons (CS) modified gravity [1], which has recently been used to propose an explanation to the cosmic baryon asymmetry [2] and polarization in the cosmic microwave background (CMB) [3]. This extension has also attracted some recent interest because it might allow for a test of a wide class of string theories [4].

CS modified gravity introduces a well-motivated correction to the Einstein-Hilbert action: the product of a parity-violating, *Pontryagin term* [59] and a scalar field or *CS coupling function* that controls the strength of the correction. Such a modification has deep roots in the standard model, since chiral, gauge and gravitational anomalies possess Pontryagin-like structures. CS modified gravity is also motivated by string theory and loop quantum gravity [5]. In the former, it arises through the Green-Schwarz mechanism [6], as an anomaly-canceling term. In fact, the CS term is a requirement of all model-independent extension of 4-dimensional compactifications of string theory [7].

Some theoretical aspects of CS modified gravity have recently been investigated. Cosmological studies have been carried out in [3, 8, 9] in connection with the Cosmic Microwave Background radiation and in [2, 10] in the context of leptogenesis. Gravitational wave solutions have also been studied in [1, 11, 12, 13], where they were found to possess amplitude birefringence, possibly leading to a test of the theory [4]. Weak-field solutions for spinning objects in CS modified gravity have been studied in [12, 13], leading to a prediction of gyromagnetic precession that differs from GR, and later to a constraint on the magnitude of the CS correction [14]. Further discussion on these issues and others related can be found in [9, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] and references therein.

Solutions to the CS modified field equations that represent interesting physical configurations have also attracted attention. Apart from first formalizing the theory, Jackiw and Pi [1] also showed that the Schwarzschild and gravitational plane-wave spacetimes remain solutions in CS modified gravity. The Reissner-Nordstrom and Friedman-Robertson-Walker metrics were also found to persist in the theory [24]. On the perturbative front, [12, 13] found a weak-field solution for orbiting, spinning objects, while [25] showed that the weak-field limit of the Kerr metric remained a CS solution to first order in the metric perturbation for a specific choice non-canonical of the CS coupling. Exact solutions that represent spinning objects in CS modified gravity have so far only been studied in [29], where it was suggested that these exist provided the gravitomagnetic sector of the metric is non-vanishing or that stationarity is broken.

In this paper, we study perturbations of a Schwarzschild black hole (BH) in CS modified gravity [60]. We begin by showing that Birkhoff's theorem – the statement that the Schwarzschild solution is the only vacuum, spherically-symmetric solution of the theory– persists in the modified theory for a wide class of coupling functions. After decomposing the metric perturbations in spherical harmonics, we find equations that govern the behaviour of each harmonic. In GR, the di-

vergence of the field equations (both at the perturbative and non-perturbative levels) implies energy-momentum conservation and, in the absence of matter, it becomes an identity that ensures diffeomorphism invariance. In CS gravity, however, the divergence of the field equations (again at the perturbative and non-perturbative levels) leads to a constraint, the so-called Pontryagin constraint, which ensures that the theory remains diffeomorphic invariant and that the Strong Equivalence Principle holds. This constraint imposes a restriction on the class of metrics that can be solutions of the theory, specially in vacuum [29]. At the perturbative level, the Pontryagin constraint is automatically satisfied for a flat background, but we show that for a BH background it forces the Cunningham-Price-Moncrief (CPM) master function, associated with axial perturbations (perturbations of odd parity), to vanish. Such a strong restriction on the possible perturbative modes a BH is allowed to possess is a distinctive feature of CS gravity.

After investigating these preliminary issues, we concentrate on the study of general perturbations of a Schwarzschild spacetime [61]. We find that the CS modification introduces new terms into the field equations that results in a mixing of polar and axial parity modes. Therefore, in CS modified gravity modes with different parity do not decouple, as is the case in GR. Moreover, the new terms contain third-order derivatives that change the basic structure of the partial differential equations that describe the behavior of the metric perturbations. In particular, the Pontryagin constraint constitutes a new equation for the metric perturbations, which in general leads to an overdetermined system. A priori, it remains unclear whether this system has non-trivial solutions, *i.e.* whether the entire set of field equations is compatible. This paper shows that at least a wide class of BH oscillation modes are forbidden in the modified theory for a large family of CS coupling functions. More specifically, we show that it is not possible to have either pure axial or pure polar oscillations of a BH in CS gravity for a wide class of coupling functions.

We end with a discussion of possible routes to extend CS modified gravity such that generic BH oscillations are allowed. A promising route is to allow the CS coupling function to be a dynamical quantity. In this case, the Pontryagin constraint is balanced by the equation of motion of the scalar field, thus preventing the CPM function to identically vanish. Moreover, the field equations are modified by the introduction of a stress-energy tensor for the CS scalar field and by terms describing perturbations of the scalar field. The extended set of field equations, which has a certain ambiguity encoded in the potential of the CS scalar field, is no longer overdetermined, and hence, it provides a suitable framework to study the modified dynamics of BH oscillations.

This paper is divided as follows: Sec. II describes the basics of CS modified gravity; Sec. III establishes Birkhoff's theorem for a family of CS coupling functions; Sec. IV begins by describing the basics of Schwarzschild BH perturbation theory, and then discusses the consequences of the linearized Pontryagin constraint and the structure of the modified field equations; Sec. V investigates perturbations of canonical CS gravity, including the impossibility of having either purely polar or purely axial perturbative modes; Sec. VI points to directions in which the theory could be extended to bypass the Pontryagin constraint and allow for generic oscillations of a Schwarzschild BH; Sec. VII concludes and points to future work.

We use the following conventions throughout this work. Greek letters and a semicolon are used to denote indices and covariant differentiation respectively on the 4-dimensional spacetime. In some cases, we denote covariant differentiation with respect to a Schwarzschild background metric by $\bar{\nabla}_\mu$. Partial differentiation of a quantity Q with respect to the coordinate x^μ is denoted as $\partial_{x^\mu} Q$ or $Q_{,\mu}$. Symmetrization and antisymmetrization is denoted with parenthesis and square brackets around the indices respectively, such as $A_{(ab)} := [A_{ab} + A_{ba}]/2$ and $A_{[ab]} := [A_{ab} - A_{ba}]/2$. We use the metric signature $(-,+,+,+)$ and geometric units in which $G = c = 1$.

II. INTRODUCTION TO CHERN-SIMONS MODIFIED GRAVITY

In this section we describe the basics of CS modified gravity as it was introduced by Jackiw and Pi [1]. (more details can be found in [1, 2, 29] and references therein). In this paper, we shall be mainly concerned with this formulation [1], but we will also discuss possible extensions later on.

The CS extension of the GR action is given by [14]

$$S_{\rm CS} \ := \ \kappa \int d^4x \sqrt{-{\rm g}} \left[R + \mathcal{L}_{\rm mat} - \frac{1}{4} \theta \, {}^*R \, R \right] , \quad (1)$$

where $\kappa = 1/(16\pi)$, g is the determinant of the spacetime metric $\mathrm{g}_{\mu\nu}$, R stands for the Ricci scalar and $\mathcal{L}_{\rm mat}$ is the Lagrangian density of the matter fields. We recognize the first two terms as the Einstein-Hilbert action in the presence of matter, while the last term is the CS modification, defined via

$${}^*R \, R := R_{\alpha\beta\gamma\delta} \, {}^*R^{\alpha\beta\gamma\delta} = \frac{1}{2} R_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\mu\nu} R^{\gamma\delta}{}_{\mu\nu} \, , \quad (2)$$

where the asterisk denotes the dual tensor, constructed using the 4-dimensional Levi-Civita tensor $\eta^{\alpha\beta\mu\nu}$ [62]. The strength of the CS correction is controlled by the scalar field θ, which we shall refer to as the *CS coupling function* or *CS scalar field*. The so-called *canonical* choice of θ corresponds to [1]

$$\theta_{\rm can} := [t/\mu, 0, 0, 0] \, , \quad (3)$$

where μ is a dimensional parameter.

As usual, we obtain the (modified) field equations by varying the action with respect to the metric, yielding

$$G_{\mu\nu} + C_{\mu\nu} \; = \; 8\pi T^{\rm mat}_{\mu\nu}, \tag{4}$$

where $T^{\rm mat}_{\mu\nu}$ is the matter stress-energy tensor. We shall refer to the quantity $C_{\mu\nu}$ as the C-tensor, defined as

$$C_{\mu\nu} := C^{(1)}_{\mu\nu} + C^{(2)}_{\mu\nu}, \tag{5}$$

where

$$C^{(1)}_{\mu\nu} \; := \; \theta_{;\sigma} \eta^{\sigma}{}_{(\mu}{}^{\alpha\beta} R_{\nu)\beta;\alpha}\,, \tag{6}$$

$$C^{(2)}_{\mu\nu} \; = \; \theta_{;\sigma\tau}\, {}^{*}R^{\sigma}{}_{(\mu}{}^{\tau}{}_{\nu)}\,. \tag{7}$$

The spatial sector of the C-tensor reduces to the 3-dimensional Cotton tensor in some symmetric cases [63]. For this reason, the quantity $v_\mu := \theta_{;\mu}$ is sometimes referred to as the embedding coordinate, since it embeds the 3-dimensional CS theory into a 4-dimensional spacetime.

Diffeomorphism invariance is preserved provided an additional equation is satisfied. This equation can be obtained by computing the covariant divergence of the modified field equations

$$C_{\mu\nu;}{}^{\mu} = \frac{1}{8} v_\nu\, {}^{*}R\,R = 8\pi T^{\rm mat;\mu}_{\mu\nu} \tag{8}$$

In vacuum, $T^{\rm mat}_{\mu\nu} = 0$ and thus we are left with the so-called Pontryagin constraint

$${}^{*}R\,R = 0\,, \tag{9}$$

which is an additional equation that the metric tensor has to satisfy.

III. BIRKHOFF'S THEOREM IN CHERN-SIMONS MODIFIED GRAVITY

Birkhoff's theorem states that the most general, spherically symmetric solution to the vacuum Einstein equations is the Schwarzschild metric. In CS modified gravity, this is not necessarily the case because the C-tensor modifies the field equations. In [1] it was shown that the Schwarzschild solution persists in CS modified gravity, but this does not necessarily imply that Birkhoff's theorem holds. Let us then study Birkhoff's theorem in CS modified gravity in more detail.

The line element of a general, spherically-symmetric spacetime can be written as a warped product of two 2-dimensional metrics [30, 31]: a Lorentzian one, g_{AB} $(A,B,\ldots,H = t,r)$, and the unit two-sphere metric, Ω_{ab} $(a,b,\ldots,h = \theta,\phi)$. This line element takes the following $2+2$ form:

$$g_{\mu\nu} dx^\mu dx^\nu = \mathrm{g}_{AB}(x^C) dx^A dx^B + r^2(x^A) \Omega_{ab}(x^c) dx^a dx^b\,, \tag{10}$$

where the warped factor is the square of the area radial coordinate r. Covariant differentiation on the Lorentzian manifold is denoted by a bar while on the 2-sphere is denoted by a colon. The case of the Schwarzschild metric in Schwarzschild coordinates in given by:

$$\mathrm{g}_{AB} dx^A dx^B = -f dt^2 + f^{-1} dr^2\,, \quad f = 1 - \frac{2M}{r}\,, \tag{11}$$

where M is the BH mass.

The Cotton tensor associated with this metric vanishes identically if the scalar field has the following form [64]:

$$\theta = \bar\theta(x^A) + r(x^A)\, \Theta(x^a)\,. \tag{12}$$

This functional form is invariant under coordinate changes that leave the $2+2$ structure of the metric [Eq. (10)] invariant, i.e. coordinate transformations: $x^A \;\to\; \tilde{x}^A = \tilde{x}^A(x^B)$, $x^a \;\to\; \tilde{x}^a = \tilde{x}^a(x^b)$, and transformations on the unit two-sphere. Moreover, one can show that the two pieces of the C-tensor shown in Eqs. (6) and (7) vanish independently for the family of scalar fields given in Eq. (12). For most of this paper, we will consider the particular case in which the scalar field also respects the spherical symmetry of the background:

$$\theta = \bar\theta(x^A)\,. \tag{13}$$

Let us comment some more on this result and place it in context. The CS scalar field θ is usually assumed to depend only on *cosmic* time, $\theta = \theta(t)$. This assumption has its roots in the work of Jackiw and Pi [1], who further imposed that $\dot\theta = d\theta/dt$ be constant, such that time-translation symmetry and space-time reparameterization of the spatial variables be preserved. With these assumptions, they showed that CS gravity can be interpreted as a 4-dimensional generalization of 3-dimensional Cotton gravity, and that the Schwarzschild and Friedman-Robertson-Walker solutions persist in the modified theory. Later on, Smith, *et. al.* [14] argued that the CS scalar field could represent some quintessence field that enforces the arrow of time associated with cosmic expansion. In this paper we have considered the most general spherically-symmetric spacetime and we have written its line element in the $2+2$ form that follows from its warped geometric structure. This form of the metric is invariant under general coordinate transformations in the two 2-dimensional manifolds, associated with this $2+2$ structure. Therefore, it is not too surprising that if Birkhoff's theorem is satisfied for the canonical choice of CS scalar field, it is also satisfied for fields that depend arbitrarily on t and r.

IV. PERTURBATIONS OF A SCHWARZSCHILD BLACK HOLE

A. Basics

We begin with a short summary of the basics of metric perturbations about a Schwarzschild background. The

spacetime metric of a perturbed BH can be written in the form

$$\mathrm{g}_{\mu\nu} = \bar{\mathrm{g}}_{\mu\nu} + h_{\mu\nu}\,, \tag{14}$$

where $\bar{\mathrm{g}}_{\mu\nu}$ is the background Schwarzschild metric (geometric objects associated with it will have an overbar) and $h_{\mu\nu}$ is a generic metric perturbation. Thanks to the spherical symmetry of the background, we can expand the metric perturbations in (tensor) spherical harmonics. In this way, we can separate the angular dependence in the perturbative equations, which yields a much more simple system of equations: a system of 1+1 partial differential equations (PDEs) in time and in the radial area coordinate r. In addition, we can distinguish between harmonics with polar and axial parity [65], also called even and odd parity modes respectively. In GR, the perturbative field equations decouple for modes of different parity, but this may not be the case for alternative theories.

Let us then split the metric perturbation $h_{\mu\nu}$ into polar and axial perturbations, $h_{\mu\nu} = h^{\mathrm{a}}_{\mu\nu} + h^{\mathrm{p}}_{\mu\nu}$, and each of these into (tensor) spherical harmonics via

$$h^{\mathrm{a}}_{\mu\nu} = \sum_{\ell,m} h^{\mathrm{a},\ell m}_{\mu\nu}\,, \quad h^{\mathrm{p}}_{\mu\nu} = \sum_{\ell,m} h^{\mathrm{p},\ell m}_{\mu\nu}\,, \tag{15}$$

where

$$h^{\mathrm{a},\ell m}_{\mu\nu} = \begin{pmatrix} 0 & h^{\ell m}_A\, S^{\ell m}_a \\ * & H^{\ell m}\, S^{\ell m}_{ab} \end{pmatrix}\,, \tag{16}$$

$$h^{\mathrm{p},\ell m}_{\mu\nu} = \begin{pmatrix} h^{\ell m}_{AB}\, Y^{\ell m} & p^{\ell m}_A\, Y^{\ell m}_a \\ * & r^2(K^{\ell m}\, Y^{\ell m}_{ab} + G^{\ell m}\, Z^{\ell m}_{ab}) \end{pmatrix}\,, \tag{17}$$

and where asterisks denote components given by symmetry. The quantity $Y^{\ell m}$ refers to the standard scalar spherical harmonics [see [32] for conventions], while $Y^{\ell m}_a$ and $S^{\ell m}_a$ are polar and axial, vector spherical harmonics, defined only for $\ell \geq 1$ via

$$Y^{\ell m}_a \equiv Y^{\ell m}_{:a}\,, \qquad S^{\ell m}_a \equiv \eta_a{}^b\, Y^{\ell m}_b\,. \tag{18}$$

Similarly, $Y^{\ell m}_{ab}$ and $Z^{\ell m}_{ab}$ are polar, and $S^{\ell m}_{ab}$ axial, symmetric tensor spherical harmonics, defined only for $\ell \geq 2$ via

$$Y^{\ell m}_{ab} \equiv Y^{\ell m}\Omega_{ab}\,, \quad Z^{\ell m}_{ab} \equiv Y^{\ell m}_{:ab} + \frac{\ell(\ell+1)}{2} Y^{\ell m}\Omega_{ab}\,, \tag{19}$$

$$S^{\ell m}_{ab} \equiv S^{\ell m}_{(a:b)}\,. \tag{20}$$

The sign convention for the Levi-Civita tensor of the 2-sphere is: $\eta_{\theta\varphi} = \sin\theta$. All metric perturbations, scalar ($h^{\ell m}_{AB}$), vectorial ($p^{\ell m}_A$ and $q^{\ell m}_A$), and tensorial ($K^{\ell m}$, $G^{\ell m}$, and $q^{\ell m}_2$), are functions of t and r only.

In GR, the Einstein equations can be decoupled in terms of complex master functions that obey wave-like master equations. Once the master functions are constructed we can recover all remaining metric perturbations from them. For axial modes, we can use the Cunningham-Price-Moncrief (CPM) master function [33], defined by

$$\Psi^{\ell m}_{\mathrm{CPM}} = -\frac{r}{\lambda_\ell}\left(h^{\ell m}_{r,t} - h^{\ell m}_{t,r} + \frac{2}{r} h^{\ell m}_t\right)\,, \tag{21}$$

whereas for polar modes we can use the Zerilli-Moncrief (ZM) master function [34, 35]

$$\begin{aligned}\Psi^{\ell m}_{\mathrm{ZM}} &= \frac{r}{1+\lambda_\ell}\left\{K^{\ell m} + (1+\lambda_\ell)G^{\ell m}\right. \\ &+ \left.\frac{f}{\Lambda_\ell}\left[f h^{\ell m}_{rr} - r K^{\ell m}_{,r} - \frac{2}{r}(1+\lambda_\ell)p^{\ell m}_r\right]\right\}\,,\end{aligned} \tag{22}$$

where $\lambda_\ell = (\ell+2)(\ell-1)/2$ and $\Lambda_\ell = \lambda_\ell + 3M/r$. These two complex master functions are gauge invariant. In order to simplify our analysis we shall here fix the gauge by setting the following metric perturbations to zero (Regge-Wheeler gauge):

$$H^{\ell m} = 0\,, \tag{23}$$

$$G^{\ell m} = p^{\ell m}_t = p^{\ell m}_r = 0\,. \tag{24}$$

The master equations for the master functions, in the case of perturbations without matter sources, have the following wave-like structure:

$$\left[-\partial^2_{t^2} + \partial^2_{r^2_\star} - V^{\mathrm{Polar/Axial}}_\ell(r)\right]\Psi^{\ell m}_{\mathrm{CPM/ZM}} = 0\,, \tag{25}$$

where $r_\star$ is the *tortoise* coordinate $r_\star = r + 2M\ln[r/(2M)-1]$. The quantity $V^{\mathrm{Polar/Axial}}_\ell(r)$ is a potential that depends on the parity and harmonic number ℓ (its precise form can be found elsewhere. To follow our notation, see [32]).

When the master functions are known, one can use them to construct the plus and cross-polarized gravitational waveforms via

$$h_+ - i h_\times = \frac{1}{2r}\sum_{\ell\geq 2,m}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left(\Psi^{\ell m}_{\mathrm{ZM}} + i\Psi^{\ell m}_{\mathrm{CPM}}\right){}_{-2}Y^{\ell m}\,, \tag{26}$$

where ${}_{-2}Y^{\ell m}$ denotes spherical harmonics of spin weight -2 [36]. One can also compute the fluxes of energy and angular momentum emitted toward infinity and also into the BH horizon in terms of the master functions. These fluxes are evaluated using a short-wavelength approximation in the *radiation zone*, where we can introduce a well-defined gauge-invariant energy-momentum tensor for gravitational radiation [37, 38, 39]. However, the expression for this effective energy-momentum tensor depends on the structure of the field equations. In the case of CS modified gravity, given that the modification in the field equations is additive, the form of these fluxes will

be the same as in GR but with extra terms proportional to the CS scalar field and its derivatives. That is,

$$\dot{E}_{\rm GW} = \frac{1}{64\pi}\sum_{\ell\geq 2,m}\frac{(\ell+2)!}{(\ell-2)!}\left(|\dot{\Psi}^{\ell m}_{\rm CPM}|^2+|\dot{\Psi}^{\ell m}_{\rm ZM}|^2\right) + \mathcal{O}(\partial\theta,\partial^2\theta)\,, \tag{27}$$

$$\dot{L}_{\rm GW} = \frac{1}{64\pi}\sum_{\ell\geq 2,m} im\frac{(\ell+2)!}{(\ell-2)!}\left(\bar{\Psi}^{\ell m}_{\rm CPM}\dot{\Psi}^{\ell m}_{\rm CPM}+\bar{\Psi}^{\ell m}_{\rm ZM}\dot{\Psi}^{\ell m}_{\rm ZM}\right) + \mathcal{O}(\partial\theta,\partial^2\theta)\,, \tag{28}$$

where the dots here denote time differentiation.

B. Pontryagin Constraint

We have seen that diffeomorphism invariance requires an extra condition, the Pontryagin constraint [Eqs. (2)-(9)]. This condition is automatically satisfied not only by the most general spherically symmetric metric (its C-tensor vanishes), but also for linear perturbations of Minkowski spacetime. Nonetheless, this condition is not satisfied for generic perturbations of a Schwarzschild BH. In this case, at first-order, the Pontryagin constraint becomes

$$^{*}R\,R = \frac{96M}{r^6}\left[h^{\ell m}_t + \frac{r}{2}\left(h^{\ell m}_{r,t}-h^{\ell m}_{t,r}\right)\right]\ell\,(\ell+1)\,Y^{\ell m}\,. \tag{29}$$

Remarkably, the Pontryagin constraint involves only axial modes, which is a consequence of the appearance of the Levi-Civita tensor (a completely antisymmetric tensor) in the modification of the gravitational sector of the action. Perhaps even more remarkably, this specific combination of axial modes corresponds exactly to the Cunningham-Price-Moncrief master function [33] [eq. (21)], which appears in the metric waveforms [Eq. (26)] and in the fluxes of energy and angular momentum [Eq. (27) and (28)]. We can thus rewrite Eq. (29) as

$$^{*}R\,R = -\frac{24M}{r^6}\frac{(\ell+2)!}{(\ell-2)!}\Psi^{\ell m}_{\rm CPM}Y^{\ell m}\,. \tag{30}$$

Then, the Pontryagin constraint forces the CPM function to vanish for all harmonics with $\ell\geq 2$. Such a restriction does not imply that all axial perturbations necessarily vanish in CS modified gravity, but it does require that these modes satisfy the relation

$$h^{\ell m}_{r,t} = h^{\ell m}_{t,r} - \frac{2}{r}h^{\ell m}_t\,. \tag{31}$$

Such a condition seems to reduce the set of possible solutions of the perturbative vacuum field equations, which might lead to an overconstrained system. For the choice of θ in Eq. (13), we shall see in Sec. V A that this is indeed the case.

C. Structure of the Modified Field Equations

The CS modified dynamics of linear perturbations about a Schwarzschild background can be studied by expanding the metric perturbations into spherical harmonics, Eqs. (15)-(17). After introducing these expansions into the field equations

$$G_{\mu\nu} = -C_{\mu\nu}\,. \tag{32}$$

we can extract individual evolution equations for each harmonic.

The choice of the CS scalar field will determine the form of the right-hand side in Eq. (32). Since we are considering perturbations about a Schwarzschild spacetime, we must choose θ such that Birkhoff's theorem holds. In Sec. III, we determined that scalar fields of the form of Eq. (12) would indeed allow Birkhoff's theorem to persist in CS modified gravity. We shall here choose θ as in Eq. (13), such that this field agrees with the symmetries of the background, namely

$$\theta = \bar{\theta}(t,r)\,, \tag{33}$$

where t and r are here the Schwarzschild time and radial coordinates. This family of CS coupling functions encompasses the canonical choice: $\bar{\theta} = t/\mu$.

The structure of the linearized field equations is analyzed by first looking at the harmonic decomposition of the Einstein tensor and the C-tensor. The structure of the former is well-known from the study of perturbations of non-rotating BHs in GR, and is given by

$$G^{\ell m}_{AB} = \mathcal{G}^{\ell m}_{AB}[\boldsymbol{U}^{\ell m}_{\rm Polar}]\,Y^{\ell m}\,, \tag{34}$$

$$G^{\ell m}_{Aa} = \mathcal{G}^{\ell m}_{A}[\boldsymbol{U}^{\ell m}_{\rm Polar}]\,Y^{\ell m}_a + \mathcal{H}^{\ell m}_{A}[\boldsymbol{U}^{\ell m}_{\rm Axial}]\,S^{\ell m}_a\,, \tag{35}$$

$$G^{\ell m}_{ab} = \mathcal{G}^{\ell m}[\boldsymbol{U}^{\ell m}_{\rm Polar}]\,Y^{\ell m}_{ab} + \mathcal{H}^{\ell m}[\boldsymbol{U}^{\ell m}_{\rm Polar}]\,Z^{\ell m}_{ab} + \mathcal{I}^{\ell m}[\boldsymbol{U}^{\ell m}_{\rm Axial}]\,S^{\ell m}_{ab}\,, \tag{36}$$

where $\boldsymbol{U}^{\ell m}_{\rm Polar}$ denotes the set of (ℓ,m)-polar perturbations

$$\boldsymbol{U}^{\ell m}_{\rm Polar} = (h^{\ell m}_{AB}\,, p^{\ell m}_A\,, K^{\ell m}, G^{\ell m})\,, \tag{37}$$

and $\boldsymbol{U}^{\ell m}_{\rm Axial}$ denotes the set of (ℓ,m)-axial perturbations

$$\boldsymbol{U}^{\ell m}_{\rm Axial} = (h^{\ell m}_A\,, H^{\ell m})\,. \tag{38}$$

Expressions for the coefficients $\mathcal{G}^{\ell m}_{AB}$, $\mathcal{G}^{\ell m}_{A}$, $\mathcal{G}^{\ell m}$, $\mathcal{H}^{\ell m}_{A}$, $\mathcal{H}^{\ell m}$, and $\mathcal{I}^{\ell m}$ are given in Appendix A. Clearly, polar spherical harmonics have functional coefficients that depend only on polar metric perturbations, while axial spherical harmonics have functional coefficients that depend only on axial metric perturbations. On the other hand, the harmonic structure of the C-tensor is given by

$$C^{\ell m}_{AB} = \mathcal{C}^{\ell m}_{AB}[\boldsymbol{U}^{\ell m}_{\rm Axial}]\,Y^{\ell m}\,, \tag{39}$$

$$C^{\ell m}_{Aa} = \mathcal{C}^{\ell m}_{A}[\boldsymbol{U}^{\ell m}_{\rm Axial}]\,Y^{\ell m}_a + \mathcal{D}^{\ell m}_{A}[\boldsymbol{U}^{\ell m}_{\rm Polar}]\,S^{\ell m}_a\,, \tag{40}$$

$$C^{\ell m}_{ab} = \mathcal{C}^{\ell m}[\boldsymbol{U}^{\ell m}_{\rm Axial}]\,Y^{\ell m}_{ab} + \mathcal{D}^{\ell m}[\boldsymbol{U}^{\ell m}_{\rm Axial}]\,Z^{\ell m}_{ab} + \mathcal{E}^{\ell m}[\boldsymbol{U}^{\ell m}_{\rm Polar}]S^{\ell m}_{ab}\,, \tag{41}$$

where explicit expressions for the coefficients $\mathcal{C}_{AB}^{\ell m}$, $\mathcal{C}_A^{\ell m}$, $\mathcal{C}^{\ell m}$, $\mathcal{D}_A^{\ell m}$, $\mathcal{D}^{\ell m}$, and $\mathcal{E}^{\ell m}$ are also given in Appendix A. In this case, polar spherical harmonics have functional coefficients that depend on axial perturbations, while axial spherical harmonics have functional coefficients that depend on polar perturbations. The main consequence of this fact is that in CS modified gravity modes with different parity are coupled, and hence, in general they cannot be treated separately.

The linearized field equations, after harmonic decomposition, become

$$\mathcal{G}_{AB}^{\ell m} = -\mathcal{C}_{AB}^{\ell m}\,, \qquad \mathcal{G}_A^{\ell m} = -\mathcal{C}_A^{\ell m}\,, \tag{42}$$

$$\mathcal{G}^{\ell m} = -\mathcal{C}^{\ell m}\,, \qquad \mathcal{H}^{\ell m} = -\mathcal{D}^{\ell m}\,, \tag{43}$$

$$\mathcal{H}_A^{\ell m} = -\mathcal{D}_A^{\ell m}\,, \qquad \mathcal{I}^{\ell m} = -\mathcal{E}^{\ell m}\,. \tag{44}$$

We can view these equations as the standard Einstein equations, linearized about a Schwarzschild background, with "source terms" that depend linearly on metric perturbations of opposite parity and their derivatives. Such a coupling between different parity modes is analogous to what occurs to gravitational-wave perturbations about a Minkowski background, where left- and right-polarized perturbations mix [1, 2, 40]. The intrinsic decoupling of the modified field equations into polar and axial sectors is thus lost.

V. BLACK HOLE PERTURBATION THEORY WITH A CANONICAL EMBEDDING

In order to understand the dynamics of the perturbations that derives from this theory and, in particular, the role and consequences of the Pontryagin constraint, we shall concentrate on the special case of a canonical CS coupling function, *i.e.* $\bar{\theta} = t/\mu$. This canonical coupling function leads to the canonical timelike embedding $v_\mu = [1/\mu, 0, 0, 0]$ and its acceleration $v_{\mu;\nu} = \Gamma^\sigma_{\mu\nu} v_\sigma$.

A. One-handed Perturbations

To begin with, we concentrate on perturbations with a single handedness, *i.e.* purely polar or purely axial perturbations. We shall analyze these cases separately.

1. Pure Axial Perturbations

We begin with pure axial metric perturbations, that is

$$h_{AB}^{\ell m} = p_A^{\ell m} = K^{\ell m} = G^{\ell m} = 0\,. \tag{45}$$

This conditions imply

$$\mathcal{G}_{AB}^{\ell m} = \mathcal{G}_A^{\ell m} = \mathcal{G}^{\ell m} = \mathcal{H}^{\ell m} = 0\,, \tag{46}$$

$$\mathcal{D}_A^{\ell m} = \mathcal{E}^{\ell m} = 0\,, \tag{47}$$

and hence the field equations reduce to:

$$\mathcal{H}_A^{\ell m} = 0\,, \qquad \mathcal{I}^{\ell m} = 0\,, \tag{48}$$

$$\mathcal{C}_{AB}^{\ell m} = 0\,, \qquad \mathcal{C}_A^{\ell m} = 0\,, \tag{49}$$

$$\mathcal{C}^{\ell m} = 0\,, \qquad \mathcal{D}^{\ell m} = 0\,. \tag{50}$$

Looking at the expressions of $\mathcal{C}^{\ell m}$ and $\mathcal{C}_{rr}^{\ell m}$ (see Appendix A) we see that both of them are proportional to the metric perturbation $h_r^{\ell m}$. Therefore, using Eqs. (49)-(50) we conclude that

$$h_r^{\ell m} = 0\,. \tag{51}$$

Similarly, $\mathcal{C}_{tr}^{\ell m}$ is proportional to $h_t^{\ell m}$, and thus, using Eq. (50)

$$h_t^{\ell m} = 0\,. \tag{52}$$

Since $H^{\ell m}$ is zero in the Regge-Wheeler gauge, we conclude that all axial perturbations must vanish. In summary, the Pontryagin constraint together with the coupling of opposite parity modes in the modified field equations, forbids the existence of purely polar oscillations of a Schwarzschild BH in CS modified gravity with a canonical embedding.

2. Pure Polar Perturbations

Let us now study pure polar perturbations by setting all axial modes to zero:

$$h_A^{\ell m} = 0\,. \tag{53}$$

The immediate consequences are

$$\mathcal{H}_A^{\ell m} = \mathcal{I}^{\ell m} = 0\,, \tag{54}$$

$$\mathcal{C}_{AB}^{\ell m} = \mathcal{C}_A^{\ell m} = \mathcal{C}^{\ell m} = \mathcal{D}^{\ell m} = 0\,, \tag{55}$$

and the field equations become

$$\mathcal{G}_{AB}^{\ell m} = \mathcal{G}_A^{\ell m} = \mathcal{G}^{\ell m} = \mathcal{H}^{\ell m} = 0\,, \tag{56}$$

$$\mathcal{D}_A^{\ell m} = \mathcal{E}^{\ell m} = 0\,. \tag{57}$$

From $\mathcal{H}^{\ell m} = 0$ we find that (see Appendix A)

$$h_{tt}^{\ell m} = f^2 h_{rr}^{\ell m}\,, \tag{58}$$

while, from equations $\mathcal{G}_A^{\ell m} = 0 = \mathcal{G}_{AB}^{\ell m}$ we find expressions for all first and second derivatives of $K^{\ell m}$ in terms of $h_{rr}^{\ell m}$, $h_{tr}^{\ell m}$ and its derivatives. Substituting these expressions into $\mathcal{E}^{\ell m} = 0$ leads to

$$h_{rr}^{\ell m} = 0\,, \tag{59}$$

which combined with Eq. (58) implies

$$h_{tt}^{\ell m} = 0\,. \tag{60}$$

There are now only two non-zero metric perturbations: $K^{\ell m}$ and $h_{tr}^{\ell m}$. Inserting the expressions for the derivatives of $K^{\ell m}$ into $\mathcal{D}_A^{\ell m} = 0$, we can solve the resulting

equations for $h^{\ell m}_{tr,rr}$ and $h^{\ell m}_{tr,tr}$. Using all the information we have collected so far in $\mathcal{G}^{\ell m} = 0$, one can show by direct evaluation that this equation requires that $h^{\ell m}_{tr,t} = 0$. Such a result, in combination with the previously found expression for $h^{\ell m}_{tr,tr}$, leads to

$$K^{\ell m} = 0\,. \tag{61}$$

Returning to the previous expressions for derivatives of $K^{\ell m}$ we obtain

$$h^{\ell m}_{tr} = 0\,. \tag{62}$$

We have then found that all polar perturbations vanish. We conclude that purely polar oscillations of a Schwarzschild BH are not allowed in CS modified gravity with a canonical embedding.

The results obtained for single-parity oscillations are quite surprising and raise questions about its robustness. In other words, can we expect the same conclusion if we repeat the analysis for other CS coupling functions? We have repeated this analysis for different CS coupling functions within the class $\theta = \bar{\theta}(t,r)$. The algebra involved is significantly more complicated and it requires intensive use of symbolic manipulation software [41]. In all studied cases, we have arrived at the same conclusion: CS modified gravity does not allow for single-parity BH oscillations.

3. *General Perturbations*

General perturbations are significantly more difficult to analyze, since we cannot separately study the components of the Einstein and C-tensors, as was the case for single parity oscillations, due to the particular structure of these tensors [Eqs. (42)-(44)]. Nonetheless, we can still make some general comments about the consequences of the Pontryagin constraint on general oscillations and the extent to which these are restricted.

The Pontryagin constraint unavoidingly adds an extra equation to the modified field equations, *i.e.* Eq. (31). Since this equation only involves axial perturbations, one may *a priori* think that polar modes are unaffected. The modified field equations, however, mix polar and axial modes, and thus, polar modes are also affected and restricted by the Pontryagin constraint. This situation has been clearly illustrated by the study of single-parity oscillations in the previous subsection.

The coupling of polar and axial metric perturbations in the modified field equations, together with the extra condition imposed by the Pontryagin constraint, lead to new conditions on the metric perturbations. In other words, these new conditions constitute new equations for the metric perturbations that are independent of the previous ones. For example, the components of $\mathcal{C}^{\ell m}_{AB}$ in Eq. (39) are linear combinations of the axial metric perturbations $h^{\ell m}_{A}$ (see Appendix A), which can be combined to reconstruct the CPM master function, leading to an additional equation for polar modes. Similarly, the components of $\mathcal{H}^{\ell m}_{A}$ [Eq. (35)] can also be combined to construct the CPM master function, and hence find another constraint on polar modes. Again, due to the mixing of different parity modes, these constraints on polar modes can in turn produce new constraints on axial modes. It is unclear where this chain of new constraints on the metric perturbations ends, but it is very likely that they will severely restrict the set of allowed BH oscillations.

One may think that choosing the CS coupling function θ appropriately, namely choosing $\bar{\theta}(x^A)$ and $\Theta(x^a)$ in Eq. (12), may end this chain of new constraints through cancellations in the generation of new equations. In the case $\theta = \bar{\theta}(x^A)$, if such cancellations were to occur for a certain harmonic number ℓ, they could not happen for other ℓ, as the equations depend on the harmonic number. Moreover, such cancellations would have to occur in different ways, associated with the different combinations of equations that produce new constraints, discussed in the previous paragraph. Adding the term $\Theta(x^a)$ to the CS scalar field only complicates the field equations further by inducing a coupling between perturbations with different harmonic number. Even in this case, given that the functional coefficient of $\Theta(x^a)$ is fixed (it is just r) and ℓ-independent, it seems unlikely that the coupling of harmonic modes will lead to the cancellations necessary to avoid new equations for the metric perturbations.

Our analysis suggests that the present structure of CS modified gravity does not seem to allow for generic BH oscillations. In particular, we have shown that for certain choices of the CS scalar field, single-parity perturbations are not allowed. For general perturbations, and with the help of computer algebra, we have analyzed the equations as discussed above and found that cancellations in the generation of new conditions on the metric perturbations are very unlikely. Therefore, the present set up for CS modified gravity does not seem to allow for the study of generic oscillations of non-rotating BHs. Such a result is reminiscent to that found in GR when additional restrictions are imposed on the metric tensor. An example of this can be found in the study of relativistic cosmological dynamics [42, 43]

Although the modified theory seems too restrictive, it is possible to develop an extension where the aforementioned constraints are avoided, while keeping the main characteristics of the modified theory untouched. Two distinctive features of the non-extended theory are the coupling of different parity modes and the Pontryagin constraint. While the former can be preserved by extensions of the modified theory, the latter must be relaxed to prevent the vanishing of the CPM master function. Since, by assumption, any extension should introduce small modifications to CS gravity, the CPM master function would be forced to be small but not quite zero. If this is the case, such extensions would have strong implications in the energy flux and, in particular, in the angular momentum flux of gravitational waves [Eqs. (27) and (28)]. These arguments motivate the study of BH

oscillations in extensions of CS modified gravity, which we shall explore in the next section.

VI. BEYOND THE CANNON

The strong restrictions obtained on the dynamics of BH oscillations thus far can be bypassed provided we considered extensions of CS modified gravity beyond the canon. One such possibility is to consider a more general CS coupling function that, for example, is not spherically symmetric. However, as we have argued before, it does not seem likely that such a modification would avoid the generation of new restrictions on the metric perturbations. Another more interesting possibility is to extend the action of Eq. (1) to allow for the dynamical evolution of the CS scalar field. Such a route is particularly promising because it weakens the Pontryagin constraint, as we shall see in this section. However, a certain amount of arbitrariness is inherent to such a route, encoded in the choice of the scalar field action and, in particular, its potential. We shall discuss this and other issues in the remaining of this section.

A. Extended CS Modified Gravity

Until now, we have treated the CS coupling function as a non-dynamical quantity. Recently, Smith, et. al. [14] added a kinetic and a potential term to the action, which they found did not contribute to their weak-field analysis. These additional terms are of the form

$$S_{\rm ext} = S_{\rm CS} + \kappa \int d^4x \sqrt{-g} \left[\frac{1}{2} \theta_{,\mu} \theta^{,\mu} - V(\theta) \right] , \qquad (63)$$

where $V(\theta)$ is some potential for the CS scalar field.

The variation of the extended action with respect to the metric and scalar field yields the equations of motion of the extended theory

$$\begin{aligned} G_{\mu\nu} + C_{\mu\nu} &= 8\pi \left(T_{\mu\nu}^{\rm mat} + T_{\mu\nu}^{\theta} \right) , && (64) \\ \square\theta &= -\frac{dV}{d\theta} - \frac{1}{4} {}^{*}R\, R \,, && (65) \end{aligned}$$

where the D'Alambertian of any scalar can be computed from

$$\square\theta = \frac{1}{\sqrt{-g}} \left[\sqrt{-g}\; g^{\mu\nu}\; \theta_{,\nu} \right]_{,\mu} , \qquad (66)$$

and where the stress-energy tensor of the scalar field is [44]

$$T_{\mu\nu}^{\theta} = \theta_{,\mu}\theta_{,\nu} - \frac{1}{2} g_{\mu\nu} \theta_{;\sigma} \theta_{;}^{\sigma} - g_{\mu\nu} V(\theta) \,. \qquad (67)$$

In this extension of CS modified gravity, the Pontryagin constraint is replaced by an equation of motion for the CS scalar field, Eq. (65), where the quadratic curvature scalar ${}^{*}R\,R$ plays the role of a driving force. Moreover, looking at the field equations for the metric, we realize that they are not only sourced by the matter fields but also by the CS scalar field through its energy-momentum content. In the next subsection we shall study what consequences this modification imposes on the satisfaction of Birkhoff's theorem and the Pontryagin constraint.

B. Birkhoff's Theorem in the Extended Theory

The simple derivation of Birkhoff's theorem in Sec. III is now modified by the extended action. For a CS scalar field of the form of Eq. (12) and the line element of Eq. (10), the C-tensor vanishes, which now leads to

$$G_{\mu\nu} = 8\pi T_{\mu\nu}^{\theta}. \qquad (68)$$

If the metric of Eq. (10) is a solution to Einstein's equations, as is the case for the Schwarzschild metric, the stress-energy tensor of the scalar field must vanish. For the choice of θ in Eq. (33) and the Schwarzschild metric, this implies the following conditions:

$$\begin{aligned} 0 &= \theta_{,t}\theta_{,r}, \\ 0 &= \frac{1}{2}\theta_{,t}^2 + \frac{f^2}{2}\theta_{,r}^2 \pm f V(\theta), && (69) \end{aligned}$$

where f is given in Eq. (11). The only solution to this system is the trivial one: $V = 0$ and $\theta =$ const., which reduces CS theory to GR. However, if we treat θ as a small quantity, as it is suggested by the different physical scenarios that motivate CS modified gravity, then Birkhoff's theorem holds to $\mathcal{O}(\theta^2)$ provided V is at least of the same order. An example of such a potential is a mass term $V = m\theta^2$, typical of scalar interactions.

The results of this subsection also apply to more general CS coupling functions and background metrics. In fact, these results hold for any line element that represents a general, spherically symmetric spacetime, *i.e.* Eq. (10). Moreover, they also hold for the most general CS field that leads to a vanishing C-tensor, *i.e.* Eq. (12), because the kinetic sector of $T_{\mu\nu}^{(\theta)}$ is always quadratic in θ.

C. BH perturbations in the Extended Theory and the Pontryagin constraint

In the extended theory, the Pontryagin constraint is replaced by a dynamical equation for the CS scalar field with a purely gravitational driving term. Since now there is no Pontryagin constraint, there is no *a priori* reason for the equations to disallow general BH oscillations. In fact, as we shall see in this subsection, the extended theory leads to a system of 11 PDEs for 11 dynamical variables $(\theta, h_{\mu\nu})$.

But can we treat θ as a perturbation? As we have just seen, Birkhoff's theorem holds only to linear order in θ,

for a wide class of potentials $V(\theta)$. On the one hand, in most string-theory scenarios that necessitate the CS correction [10, 14, 23], the scalar field θ is proportional to the string scale. In such cases, θ is much smaller than any metric perturbation and Birkhoff's theorem holds. Nonetheless, even within such frameworks, the CS correction could be enhanced by couplings to non-perturbative, string theoretical degrees of freedom (*i.e.* gravitational instantons) [45]. Moreover, there are some theoretical frameworks where the string coupling $g_{\rm s}$ vanishes at late times [46, 47, 48, 49, 50, 51, 52, 53, 54, 55], in which case a larger coupling is permissible.

From this discussion, we cannot necessarily assume that the magnitude of the scalar field is smaller or of the same order as the metric perturbations. In fact, these perturbations and the scalar field act on completely independent scales. Therefore, all we can assume is that θ and $|h_{ab}|$ are both *independently* smaller than the background, which is enough to justify the use of perturbation theory.

We shall thus consider a two-parameter (bivariate) perturbative expansion of CS gravity. One perturbative parameter shall be associated with the metric perturbations, ϵ, and the other with the scalar field, τ. The metric in Eq. (14) and the scalar field can then be rewritten as

$$\mathrm{g}_{\mu\nu} = \bar{\mathrm{g}}_{\mu\nu} + \epsilon\, h_{\mu\nu}\,, \tag{70}$$

$$\begin{aligned} \theta &= \tau(\bar{\theta} + \epsilon\, \delta\theta) \\ &= \tau\bar{\theta} + \tau\epsilon \sum_{\ell\geq 1, m} \tilde{\theta}^{\ell m}\, Y^{\ell m}\,, \end{aligned} \tag{71}$$

where $\bar{\theta}$ satisfies Birkhoff's theorem [Eq. (12)] and respects the spherical symmetry of the background. The quantities $\tilde{\theta}^{\ell m} = \tilde{\theta}^{\ell m}(x^A)$ are harmonic coefficients of the scalar field perturbations, associated with the BH oscillations. There are no $\ell = 0$ modes in the sum of Eq. (71) because they can always be absorbed in the monopole term $\bar{\theta}$.

The equations of motion for the metric perturbation and the scalar field now become formal bivariate expansions in $\epsilon \ll 1$ as well as $\tau \ll 1$. The modified field equations to zeroth order in ϵ are simply equations for the background metric, which are automatically satisfied to this order by Birkhoff's theorem. The equation of motion for the scalar field to the same order becomes

$$\tau\, \bar{\Box}\theta = \tau \bar{\mathrm{g}}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \theta = 0\,. \tag{72}$$

In Eq. (72), and in all remaining equations, the overhead bars on any quantity are to remind us that these quantities are to be evaluated with respect to the background metric $\bar{\mathrm{g}}_{\mu\nu}$. Moreover, in Eq. (72) and henceforth, we shall neglect the contribution of the potential, by assuming that it is at least of $\mathcal{O}(\tau^2)$. As discussed earlier, the potential encodes a certain arbitrariness in the extension of the CS modification, which is why we have chosen to neglect it.

The first-order equations govern the dynamics of the metric perturbation and the perturbations of the scalar field. The equation of motion for the scalar field to $\mathcal{O}(\epsilon)$ is given by

$$\begin{aligned} \epsilon \frac{1}{4} \delta({}^*R\, R) &= \epsilon\tau \left\{ \bar{\Box}\delta\theta - \left[\bar{\theta}_{,\mu\nu} + \left(\ln\sqrt{-\bar{g}}\right)_{,\mu} \bar{\theta}_{,\nu} \right] h^{\mu\nu} \right. \\ &- \left. \bar{\theta}_{,\mu} h^{\mu\nu}{}_{,\nu} + \frac{1}{2} h_{,\mu} \bar{\mathrm{g}}^{\mu\nu} \bar{\theta}_{,\nu} \right\}. \end{aligned} \tag{73}$$

In Eq. (73), $\delta({}^*R\, R)$ is the functional coefficient of ${}^*R\, R$ to $\mathcal{O}(\epsilon)$ [Eq. (30)] and $h = \mathrm{g}^{\mu\nu} h_{\mu\nu} = \bar{\mathrm{g}}^{\mu\nu} h_{\mu\nu}$. Equation (73) is harmonically decomposed in the Regge-Wheeler gauge in Eq. (B2) of Appendix B.

Similarly, the equations of motion for the metric perturbation to $\mathcal{O}(\epsilon)$ reduce to

$$\begin{aligned} \mathcal{O}(\tau^2) &= \epsilon\delta G_{\mu\nu} + \epsilon\tau \left[\bar{\theta}_{,\sigma} \bar{\eta}^{\sigma\alpha\beta}{}_{(\mu} \bar{\nabla}_\alpha \delta R_{\nu)\beta} \right. \\ &+ \left(\bar{\nabla}_\tau \bar{\theta}_{,\sigma}\right) \delta\, {}^*R^\sigma{}_{(\mu}{}^\tau{}_{\nu)} + \left(\bar{\nabla}_\tau \delta\theta_{,\sigma}\right. \\ &- \left. \left. \bar{\theta}_{,\rho} \delta\Gamma^\rho_{\sigma\tau}\right)\; {}^*\bar{R}^\sigma{}_{(\mu}{}^\tau{}_{\nu)} \right]. \end{aligned} \tag{74}$$

In Eq. (74), $\delta G_{\mu\nu}$ is the functional coefficient of $G_{\mu\nu}$ to $\mathcal{O}(\epsilon)$ (see Appendix A), and all other $\delta A^{\alpha\beta\cdots}{}_{\chi\zeta\ldots}$ stands for the coefficient to $\mathcal{O}(\epsilon)$ of any tensor $A^{\alpha\beta\cdots}{}_{\chi\zeta\ldots}$. In fact, Eq. (74) is the formal covariant expression of the perturbation of the Einstein and C-tensors to $\mathcal{O}(\epsilon)$, which we computed in Appendix A for the CS coupling function in Eq. (33), with the exception of the term proportional to $\delta\theta$. The quadratic terms on the right-hand side of Eq. (74) come from the energy-momentum tensor of the scalar field θ [see Eq. (64)]. When taking the divergence of this equation, we recover Eq. (73) only when quadratic terms in τ are taken into account because functional differentiation with respect to θ reduces the order in τ of the resulting expression by unity.

Equations (73) and (74) govern the dynamics of BH oscillations in this extended version of CS modified gravity, but how do we solve them? Although it would be useful to decouple these equations in terms of master functions as done in GR, this is not an easy task. In addition to the mixing of parity modes and the fact that the Pontryagin constraint can no longer be used to simplify equations, there is now additional terms in the modified field equations due to the perturbations of the scalar field. Moreover, these perturbations possess their own dynamics, and hence, any decoupling should involve the whole set of perturbative variables, *i.e.* $(\tilde{\theta}^{\ell m}, h^{\ell m}_{\mu\nu})$.

Nonetheless, it should be possible to numerically solve the perturbative field equations of the extended modified theory in an iterative way with $\tau \ll 1$. One such possible iterative procedure is as follows. To zeroth order in τ, the equations reduce to those of GR, which can be decoupled and solved numerically with standard methods. The numerical result can then be reinserted in the field equations to first order in τ. These equations can now be decoupled in exactly the same way and will contain source terms determined by the zeroth order solution. This problem will be tackled in a future publication.

Before concluding, let us make some general remarks about the consequences of extending CS modified gravity through such kinetic terms. Equation (73) shows that the Pontryagin constraint has become an evolution equation for the perturbations of the scalar field, $\delta\theta$. Together with Eq. (74), this evolution equation constitutes a system of PDEs for the perturbative variables $(\tilde{\theta}^{\ell m}, h_{\mu\nu}^{\ell m})$ that could in principle allow for generic oscillations. However, since $\tau \ll 1$, the magnitude of the CPM master function is forced to be small [of $\mathcal{O}(\tau)$]. Otherwise, the scalar field would lead to an amplification of the CS correction to levels forbidden by solar system tests [14]. The extended theory thus relaxes the vanishing of the CPM function and replaces it by somewhat of a suppression of radiative axial modes.

Such an effect may lead to important observational consequences. In particular, the dynamics of astrophysical systems, where axial modes contribute significantly to the gravitational-wave emission, would be greatly modified. In such cases, Eqs. (27) and (28) suggest that the flux of energy emitted would be dominated by polar modes and the flux of angular momentum would be very small, since it would be linear in τ. Such a flux suppression is in contrast to predictions of GR, where the gravitational wave emission of angular momentum is known to be large (approximately 14% of the initial ADM angular momentum for quasi-circular BH mergers [56]). Consequently, the dynamics of gravitational wave sources in the radiation-reaction dominated phase should be quite different in CS modified gravity relative to GR, thus allowing for gravitational wave tests of the extended theory.

VII. CONCLUSIONS

We have studied Schwarzschild BH perturbation theory in CS modified gravity. We began by showing that Birkhoff's theorem (the statement that the Schwarzschild solution is the only vacuum, spherically-symmetric solution of the theory) holds in the modified theory for a wide class of CS coupling functions. We then decomposed the metric perturbations into tensor spherical harmonics and found the linearized modified field equations that determine their behavior. The divergence of these equations led to the linearized Pontryagin constraint, which imposes a restriction on axial metric perturbations –the CPM master function has to vanish.

Once these preliminary issues were studied, we focused on the general structure of the metric perturbations in CS modified gravity. We found that the modified theory adds new terms to the field equations that couple perturbations with polar and axial parity. Moreover, due to the restrictions imposed by the Pontryagin constraint, we find that in general the system of equations is overdetermined. Whether the entire set of equations is compatible remains unclear, but for a wide class of initial physical conditions, we found that linear BH oscillations are not allowed in CS modified gravity. Specifically, we showed that pure axial or pure polar oscillations are disallowed for a wide class of coupling functions.

Possible extensions of the modified theory that would allow for generic BH oscillations were also discussed. In particular, we investigated the possibility of providing the CS coupling function with dynamics. In other words, the inclusion of a kinetic and potential term in the action led to the replacement of the Pontryagin constraint with an equation of motion for the coupling function. This route then lifts the vanishing restriction of the Pontryagin constraint and imposes a smallness condition on the CPM function.

The extended CS modified framework thus allows for generic BH oscillations but it imposes an important smallness restriction on axial perturbation that could lead to astrophysical observables. In particular, we saw that such a suppression of axial modes would lead to a significant decrease in the magnitude of energy, linear and angular momentum carried by gravitational radiation relative to GR. Such a decrease in gravitational wave intensity would have important consequences in the dynamics of compact object inspirals, specially in the radiation-reaction dominated phase.

Future observations of the ringdown signal in binary BH mergers could be used to test and constrain CS modified gravity. For such studies, the results found in this paper would be critical in order to determine the quasi-normal frequency spectrum of perturbations. Future work could concentrate on such a spectrum, by numerically studying the linearized and harmonically-decoupled field equations presented here. Moreover, semi-analytic studies might also be possible through the close-limit approximation. Only through detailed studies of all aspects of the modified theory and its links to experimental observations will we be able to determine its viability.

Acknowledgments

We would like to thank Eric Poisson and Jorge Pullin for encouragement to study perturbations of black holes in the context of CS gravity. We would also like to thank Ted Jacobson, Stephon Alexander, Daniel Grumiller, Ben Owen, and Richard O'Shaughnessy for discussions. Most of our calculations used the computer algebra systems MAPLE v.11 in combination with the GRTensorII package [41].

NY would also like to thank Ben Owen for his ongoing support. CFS acknowledges support from the Natural Sciences and Engineering Research Council of Canada during the first stages of this work. Presently, he is supported by the Ramon y Cajal Programme of the Ministry of Education and Science of Spain and by a Marie Curie International Reintegration Grant (MIRG-CT-2007-205005/PHY) within the 7th European Community Framework Programme. NY acknowledges the support of the Center for Gravitational Wave Physics

funded by the National Science Foundation under Cooperative Agreement PHY-01-14375, support from NSF grant PHY-05-55-628, and the University of Guelph for hospitality during a visit in which this work was started.

APPENDIX A: EXPLICIT EXPRESSIONS FOR THE PERTURBED C- AND EINSTEIN TENSORS

Here we provide explicit expressions in Schwarzschild coordinates for the coefficients of the expansion in spherical harmonics of the Einstein and C-tensors. In these expressions, the Pontryagin constraint [Eq. (9)] has been used. For simplicity we omit the superscripts $(\ell\,,m)$. The components of the harmonically decomposed Einstein tensor read

$$\mathcal{G}_{tt} = -f^2 K_{,rr} - f\left(\frac{f'}{2}+\frac{3f}{r}\right)K_{,r} + \frac{(\ell+2)(\ell-1)f}{2r^2}K + \frac{f^3}{r}h_{rr,r} + \frac{f^2}{r^2}\left[\frac{\ell(\ell+1)}{2}+f+2rf'\right]h_{rr}\,, \tag{A1}$$

$$\mathcal{G}_{tr} = -K_{,tr} + \frac{1}{2f}\left(f'-\frac{2f}{r}\right)K_{,t} + \frac{f}{r}h_{rr,t} + \frac{\ell(\ell+1)}{2r^2}h_{tr}\,, \tag{A2}$$

$$\mathcal{G}_{rr} = -\frac{1}{f^2}K_{,tt} + \frac{1}{2f}\left(f'+\frac{2f}{r}\right)K_{,r} - \frac{(\ell+2)(\ell-1)}{2r^2 f}K - \frac{1}{rf}h_{tt,r} + \frac{1}{r^2 f^2}\left[\frac{\ell(\ell+1)}{2}+rf'\right]h_{tt} + \frac{2}{rf}h_{tr,t} - \frac{1}{r^2}h_{rr}\,, \tag{A3}$$

$$\mathcal{G}_{t} = -\frac{1}{2}K_{,t} + \frac{f}{2}h_{tr,r} - \frac{f}{2}h_{rr,t} + \frac{f'}{2}h_{tr}\,, \tag{A4}$$

$$\mathcal{G}_{r} = -\frac{1}{2}K_{,r} + \frac{1}{2f}h_{tt,r} - \frac{1}{4f^2}\left(f'+\frac{2f}{r}\right)h_{tt} - \frac{1}{2f}h_{tr,t} + \frac{1}{4}\left(f'+\frac{2f}{r}\right)h_{rr}\,, \tag{A5}$$

$$\mathcal{H}_{t} = \frac{(\ell+2)(\ell-1)}{2r^2}h_{t}\,, \tag{A6}$$

$$\mathcal{H}_{r} = \frac{(\ell+2)(\ell-1)}{2r^2}h_{r}\,, \tag{A7}$$

$$\begin{aligned}\mathcal{G} \;=\; & -\frac{r^2}{2f}K_{,tt} + \frac{r^2 f}{2}K_{,rr} + \frac{r^2}{2}\left(f'+\frac{2f}{r}\right)K_{,r} - \frac{r^2}{2}h_{tt,rr} + \frac{r^2}{4f}\left(f'-\frac{2f}{r}\right)h_{tt,r}\\ & + \frac{r^2}{2f}\left[\frac{\ell(\ell+1)}{2r^2}+f''+\frac{f'}{r}-\frac{f'^2}{2f}\right]h_{tt} + r^2 h_{tr,tr} + \frac{r^2}{2f}\left(f'+\frac{2f}{r}\right)h_{tr,t} - \frac{r^2}{2}h_{rr,tt}\\ & - \frac{r^2 f}{4}\left(f'+\frac{2f}{r}\right)h_{rr,r} - \frac{r^2 f}{2}\left[\frac{\ell(\ell+1)}{2r^2}+f''+\frac{3f'}{r}+\frac{f'^2}{2f}\right]h_{rr}\,,\end{aligned} \tag{A8}$$

$$\mathcal{H} = \frac{1}{2f}\left(h_{tt}-f^2 h_{rr}\right)\,, \tag{A9}$$

$$\mathcal{I} = -\frac{1}{f}\left(h_{t,t}-f^2 h_{r,r}-ff'h_{r}\right)\,. \tag{A10}$$

while the components of the harmonically decomposed C-tensor read

$$\mathcal{C}_{tt} = -\frac{(l+2)!}{(l-2)!}\frac{f}{2r^4}\bar{\theta}_{,r}h_t\,, \tag{A11}$$

$$\mathcal{C}_{tr} = -\frac{(\ell+2)!}{(\ell-2)!}\frac{1}{4r^4 f}\left(\bar{\theta}_{,t}h_t + f^2\bar{\theta}_{,r}h_r\right)\,, \tag{A12}$$

$$\mathcal{C}_{rr} = -\frac{(\ell+2)!}{(\ell-2)!}\frac{\bar{\theta}_{,t}}{2r^4 f}h_r\,, \tag{A13}$$

$$\mathcal{C}_t = -\frac{(\ell+2)(\ell-1)}{4r^2}\left\{\bar{\theta}_{,t}\left(h_{t,r} - \frac{2h_t}{r}\right) + f^2\bar{\theta}_{,r}\left[h_{r,r} - \frac{2h_r}{rf}\left(1-\frac{4M}{r}\right) - \frac{2h_{t,t}}{f^2}\right] + f^2\bar{\theta}_{,rr}h_r - \bar{\theta}_{,tr}h_t\right\}\,, \tag{A14}$$

$$\mathcal{C}_r = -\frac{(\ell+2)(\ell-1)}{4r^3 f^2}\left[\bar{\theta}_{,t}\left(-rh_{t,t} + 2rf^2 h_{r,r} + rff'h_r - 2f^2 h_r\right) - \bar{\theta}_{,r}\left(-rf^2 h_{t,r} + rff'h_t\right) - \bar{\theta}_{,tt}rh_t + \bar{\theta}_{,tr}f^2 rh_r\right]\,, \tag{A15}$$

$$\begin{aligned}
\mathcal{D}_t \;=\;& \bar{\theta}_{,t}\left(\frac{1}{4}K_{,tr} - \frac{1}{4f}\left(f' - \frac{2f}{r}\right)K_{,t} + \frac{f}{4}h_{tr,rr} + \frac{f'}{4}h_{tr,r} - \frac{f}{2r}h_{rr,t}\right.\\
&-\left.\frac{1}{4}\left\{-f'' + \frac{f'^2}{f} + \frac{2}{r^2}\left[\frac{\ell}{2}(\ell+1) + f - 1 - rf'\right]\right\}h_{tr} - \frac{f}{4}h_{rr,tr}\right)\\
&+\bar{\theta}_{,r}\left\{\frac{Kf}{4r^2}(\ell+2)(\ell-1) + \frac{h_{tt}}{8r^2 f}\left[2f\ell(\ell+1) - 1 - 2f + f^2\right] - \frac{h_{rr}f}{8r^2}\left(1 - 6f + 3f^2\right)\right.\\
&-\left.\frac{1-3f}{8r}\left(f^2 h_{rr,r} - h_{tt,r}\right) - \frac{f}{4}\left(h_{tt,rr} - \frac{2}{r}h_{tr,t} - h_{tr,tr} + fK_{,rr}\right) - \frac{f}{4r}K_{,r}(1+f)\right\}\\
&+\bar{\theta}_{,rr}\left\{\frac{1}{8r}\left[(3f-1)h_{rr}f^2 + (1+f)h_{tt}\right] - \frac{f}{4}\left(h_{tt,r} - h_{tr,t} + fK_{,r}\right)\right\}\\
&+\bar{\theta}_{,tr}\left[\frac{1}{4}K_{,t} - \frac{f}{4}h_{rr,t} + \frac{f}{4}h_{tr,r} - \frac{h_{tr}}{4r}(3f-1)\right] - \frac{ff'''}{4}\tilde{\theta} + \frac{3M}{r^3}f\tilde{\theta}_{,r}\,,
\end{aligned} \tag{A16}$$

$$\begin{aligned}
\mathcal{D}_r \;=\;& \frac{\bar{\theta}_{,t}}{4f}\left\{-h_{rr,tt} + \frac{1}{r^2}\left[\ell(\ell+1) - 2\right](K - fh_{rr}) + h_{tr,t}\left(-\frac{2}{r} + \frac{f'}{f}\right) + h_{tr,tr} + \frac{1}{f}K_{,tt}\right\}\\
&+\bar{\theta}_{,r}\left[-\frac{1}{8r}(f'r - 2f)h_{rr,t} + \frac{1}{8f^2 r}(f'r + 2f)h_{tt,t} + \frac{1}{4f}\left(h_{tr,tt} - h_{tt,tr} - fK_{,tr}\right) + \frac{(\ell+2)(\ell-1)}{4r^2}h_{tr}\right]\\
&-\frac{\bar{\theta}_{,tt}}{4rf^2}\left(-rfh_{tr,r} + 2h_{tr}f + rfh_{rr,t} - rf'h_{tr} - rK_{,t}\right)\\
&+\bar{\theta}_{,tr}\left[\frac{1}{4f}\left(h_{tr,t} - fK_{,r} - h_{tt,r}\right) + \frac{h_{tt}}{8rf^2}(rf' + 2f) - \frac{h_{rr}}{8r}(rf' - 2f)\right] + \frac{3M}{fr^3}\tilde{\theta}_{,t}\,,
\end{aligned} \tag{A17}$$

$$\mathcal{C} = \frac{(\ell+2)!}{(\ell-2)!}\frac{1}{4r^2}\left(\bar{\theta}_{,t}h_r - \bar{\theta}_{,r}h_t\right)\,, \tag{A18}$$

$$\begin{aligned}
\mathcal{D} \;=\;& \bar{\theta}_{,t}\left\{-fh_{r,rr} - \left(\frac{3}{2}f' - \frac{f}{r}\right)h_{r,r} + \frac{1}{2}\left[\frac{\ell(\ell+1)}{r^2} - f'' + \frac{2}{r}\left(f' - \frac{f}{r}\right)\right]h_r + \frac{1}{f}h_{t,tr} - \frac{1}{2f^2}\left(f' + \frac{2f}{r}\right)h_{t,t}\right\}\\
&+\bar{\theta}_{,r}\left\{-\frac{1}{f}\left(h_{t,tt} - f^2 h_{t,rr}\right) - \frac{h_t}{2r^2}\left[4f'r + f''r^2 - 6f + \ell(\ell+1)\right] + \frac{1}{r}(f'r - 2f)h_{t,r}\right\}\\
&+\left(\bar{\theta}_{,rr} + \frac{\bar{\theta}_{,tt}}{f^2}\right)\left[-\frac{h_t}{2r}(2f + f'r) + fh_{t,r}\right] - \bar{\theta}_{,tr}\left(fh_{r,r} + \frac{h_{t,t}}{f}\right)\,,
\end{aligned} \tag{A19}$$

$$
\begin{aligned}
\mathcal{E} \;=\;& \bar{\theta}_{,t}\left[\frac{1}{2}K_{,r}+\frac{1}{2f}h_{tr,t}-\frac{f}{2}h_{rr,r}-\frac{f'}{2}h_{rr}\right]+\frac{\bar{\theta}_{,r}}{2f}\left(f^{2}h_{tr,r}-fK_{,t}-h_{tt,t}+ff'h_{tr}\right)\\
+\;& \frac{f}{2}h_{tr}\left(\bar{\theta}_{,rr}+\frac{\bar{\theta}_{,tt}}{f^{2}}\right)-\frac{\bar{\theta}_{,tr}}{2f}\left(h_{tt}+f^{2}h_{rr}\right)\,. \qquad \text{(A20)}
\end{aligned}
$$

APPENDIX B: EQUATIONS OF THE EXTENDED CS THEORY IN THE REGGE-WHEELER GAUGE

The equation for the background component of the scalar field, Eq. (72), is

$$
\left(-\partial^{2}_{t^2}+\partial^{2}_{r_\star^2}-\frac{2Mf}{r^{3}}\right)\left(r\,\bar{\theta}\right)=0\,, \qquad \text{(B1)}
$$

The divergence of the field equations, which in the non-extended theory leads to the Pontryagin constraint, now leads to the equations of motion for the scalar field. To leading order in ϵ, this equation reduces to Eq. (73), which when harmonically decomposed becomes

$$
\begin{aligned}
& \left[f\tilde{\theta}^{\ell m}_{,rr}-\frac{1}{f}\tilde{\theta}^{\ell m}_{,tt}+\frac{2}{r}\left(1-\frac{M}{r}\right)\tilde{\theta}^{\ell m}_{,r}-\frac{\tilde{\theta}^{\ell m}}{r^{2}}\ell\left(\ell+1\right)\right]-\left[f^{2}h^{\ell m}_{rr}\bar{\theta}_{,rr}+\frac{1}{f^{2}}h^{\ell m}_{tt}\bar{\theta}_{,tt}\right]\\
+\;& \left(-\frac{1}{2f^{2}}h^{\ell m}_{tt,t}-\frac{1}{2}h^{\ell m}_{rr,t}+h^{\ell m}_{tr,r}+\frac{2}{r}h^{\ell m}_{tr}\right)\bar{\theta}_{,t}+\left[-\frac{f^{2}}{2}h^{\ell m}_{rr,r}-\frac{1}{2}h^{\ell m}_{tt,r}\right.\\
+\;& \left.h^{\ell m}_{tr,t}+\frac{M}{r^{2}f}h^{\ell m}_{tt}-\frac{f}{2r}\left(3+f\right)h^{\ell m}_{rr}\right]\bar{\theta}_{,r}+fK^{\ell m}_{,r}\bar{\theta}_{,r}-\frac{K^{\ell m}_{,t}}{f}\bar{\theta}_{,t}=\frac{6M}{r^{6}}\frac{(\ell+2)!}{(\ell-2)!}\Psi^{\ell m}_{\rm CPM}\,. \qquad \text{(B2)}
\end{aligned}
$$

[1] R. Jackiw and S. Y. Pi, Phys. Rev. **D68**, 104012 (2003), gr-qc/0308071.
[2] S. H. S. Alexander, M. E. Peskin, and M. M. Sheik-Jabbari, Phys. Rev. Lett. **96**, 081301 (2006), hep-th/0403069.
[3] A. Lue, L.-M. Wang, and M. Kamionkowski, Phys. Rev. Lett. **83**, 1506 (1999), astro-ph/9812088.
[4] S. Alexander, L. S. Finn, and N. Yunes, in progress (2007).
[5] S. Alexander, in progress (2007).
[6] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory. Vol. 2: Loop Amplitides, Anomalies and Phenomenology* (Cambridge University Press, Cambridge, UK, 1987).
[7] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond* (Cambridge University Press, Cambridge, UK, 1998).
[8] M. Li, J.-Q. Xia, H. Li, and X. Zhang (2006), hep-ph/0611192.
[9] S. H. S. Alexander (2006), hep-th/0601034.
[10] S. H. S. Alexander and J. Gates, S. James, JCAP **0606**, 018 (2006), hep-th/0409014.
[11] S. Alexander and J. Martin, Phys. Rev. **D71**, 063526 (2005), hep-th/0410230.
[12] S. Alexander and N. Yunes (2007), hep-th/0703265.
[13] S. Alexander and N. Yunes, Phys. Rev. **D75**, 124022 (2007), arXiv:0704.0299 [hep-th].
[14] T. L. Smith, A. L. Erickcek, R. R. Caldwell, and M. Kamionkowski (2007), arXiv:0708.0001 [astro-ph].
[15] V. A. Kostelecky, Phys. Rev. **D69**, 105009 (2004), hep-th/0312310.
[16] T. Mariz, J. R. Nascimento, E. Passos, and R. F. Ribeiro, Phys. Rev. **D70**, 024014 (2004), hep-th/0403205.
[17] R. Bluhm and V. A. Kostelecky, Phys. Rev. **D71**, 065008 (2005), hep-th/0412320.
[18] C. Eling, T. Jacobson, and D. Mattingly (2004), gr-qc/0410001.
[19] D. H. Lyth, C. Quimbay, and Y. Rodriguez, JHEP **03**, 016 (2005), hep-th/0501153.
[20] D. Mattingly, Living Rev. Rel. **8**, 5 (2005), gr-qc/0502097.
[21] R. Lehnert (2006), gr-qc/0602073.
[22] A. J. Hariton and R. Lehnert, Phys. Lett. **A367**, 11 (2007), hep-th/0612167.
[23] S. H. Alexander, M. E. Peskin, and M. M. Sheikh-Jabbari (2007), hep-ph/0701139.
[24] D. Guarrera and A. J. Hariton (2007), gr-qc/0702029.
[25] K. Konno, T. Matsuyama, and S. Tanda, Phys. Rev. **D76**, 024009 (2007), arXiv:0706.3080 [gr-qc].
[26] W. Fischler and S. Paban (2007), arXiv:0708.3828 [hep-th].
[27] B. Tekin (2007), arXiv:0710.2528 [gr-qc].
[28] D. Grumiller and R. Jackiw (2007), arXiv:0711.0181 [math-ph].

[29] D. Grumiller and N. Yunes (2007), arXiv:0711.1868 [gr-qc].
[30] U. H. Gerlach and U. K. Sengupta, Phys. Rev. **D19**, 2268 (1979).
[31] U. H. Gerlach and U. K. Sengupta, Phys. Rev. **D22**, 1300 (1980).
[32] C. F. Sopuerta, N. Yunes, and P. Laguna, Phys. Rev. **D74**, 124010 (2006), astro-ph/0608600.
[33] C. T. Cunningham, R. H. Price, and V. Moncrief, Astrophys. J. **224**, 643 (1978).
[34] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).
[35] V. Moncrief, Ann. Phys. (N.Y.) **88**, 323 (1974).
[36] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan, J. Math. Phys. **8**, 2155 (1967).
[37] R. A. Isaacson, Phys. Rev. **166**, 1263 (1968).
[38] R. A. Isaacson, Phys. Rev. **166**, 1272 (1968).
[39] C. W. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman & Co., San Francisco, 1973).
[40] S. Alexander and J. Martin, Phys. Rev. **D71**, 063526 (2005), hep-th/0410230.
[41] *Grtensorii*, this is a package which runs within Maple but distinct from packages distributed with Maple. It is distributed freely on the World-Wide-Web from the address: `http://grtensor.org`.
[42] C. F. Sopuerta, Phys. Rev. **D55**, 5936 (1997).
[43] C. F. Sopuerta, R. Maartens, G. F. R. Ellis, and W. M. Lesame, Phys. Rev. **D60**, 024006 (1999), gr-qc/9809085.
[44] S. M. Carrol, *An introduction to general relativity, Spacetime and Geometry* (Pearson - Benjamin Cummings, San Francisco, 2003).
[45] P. Svrcek and E. Witten, JHEP **06**, 051 (2006), hep-th/0605206.
[46] R. H. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989).
[47] A. A. Tseytlin and C. Vafa, Nucl. Phys. **B372**, 443 (1992), hep-th/9109048.
[48] A. Nayeri, R. H. Brandenberger, and C. Vafa, Phys. Rev. Lett. **97**, 021302 (2006), hep-th/0511140.
[49] C.-Y. Sun and D.-H. Zhang (2006), hep-th/0611101.
[50] D. H. Wesley, P. J. Steinhardt, and N. Turok, Phys. Rev. **D72**, 063513 (2005), hep-th/0502108.
[51] R. Brandenberger, D. A. Easson, and D. Kimberly, Nucl. Phys. **B623**, 421 (2002), hep-th/0109165.
[52] T. Battefeld and S. Watson, Rev. Mod. Phys. **78**, 435 (2006), hep-th/0510022.
[53] R. H. Brandenberger, A. Nayeri, S. P. Patil, and C. Vafa (2006), hep-th/0608121.
[54] R. Brandenberger (2007), hep-th/0702001.
[55] P. Brax, C. van de Bruck, and A.-C. Davis, Rept. Prog. Phys. **67**, 2183 (2004), hep-th/0404011.
[56] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, Phys. Rev. Lett. **96**, 111101 (2006), gr-qc/0511048.
[57] D. Grumiller, *private communication*.
[58] A. Garcia, F. W. Hehl, C. Heinicke, and A. Macias, Class. Quant. Grav. **21**, 1099 (2004), gr-qc/0309008.
[59] See [29] for a discussion of this term and some physical interpretations
[60] The term"CS modified gravity" stands for the formulation introduced in [1]. Extensions of this formulation have been considered in [14, 29] and we shall discuss how these affect the analysis of this paper in the last section.
[61] BH perturbations were previously studied in [25], but only perturbative modes that resemble the weak-field limit of the Kerr metric were considered. In this paper, we consider the most general perturbations in a Schwarzschild background.
[62] In non-vacuum spacetimes, the dual of the Riemann tensor with respect to the first pair of indices does not coincide with the dual with respect to the second pair of indices. In this paper we use the second type of dual, but the results of this paper are independent of this choice. This can also be seen from the condition ${}^*R\,R = {}^*C C$ [57]
[63] In [1], $C_{\mu\nu}$ was identified with the 'Cotton tensor' because it reduces to its 3-dimensional counterpart in certain symmetric cases. However, a higher-dimensional Cotton tensor already exists [58], and its definition is different from that of Eq. (5). To avoid confusion, we refer to $C_{\mu\nu}$ as the C-tensor instead of the Cotton tensor.
[64] A similar result is found in [29]
[65] Polar and axial modes as those which, under a parity transformation, acquire a factor of $(-1)^l$ and $(-1)^{l+1}$ respectively.